Big Line Bundles

X irr. projective variety of dimension n, L a line bundle on X.

L is **big** if $\kappa(L) = n$.

Ex: • Ample $\Rightarrow$ big, but nef $\not\Rightarrow$ big (e.g. $\mathcal{O}_X$)

• $f: X \to Y$ generically finite, L ample $\Rightarrow$ f^*L big.

Rmk: Iitaka fibration Thm says that L big $\Leftrightarrow$ $\varphi_m : X \dashrightarrow Y_m$ is birational for suff. large $m \in \mathbb{N}(L)$.

Alternate def in terms of growth of global sections

X projective of dimension n. A divisor $D \subseteq X$ is big

$\Leftrightarrow$ there's a constant $C > 0$ s.t. $h^0(\mathcal{O}(mD)) \geq C \cdot m^n$

$\forall$ suff. large $m \in \mathbb{N}(D)$. $(*)$

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Pf: If X is normal, then "$\Rightarrow$" follows immediately from last time, and to get "$\Leftarrow$",

If $h^0(\mathcal{O}_X(mD)) \geq Cm^n$, then it's not smaller than $a \cdot m^k$ for any $k < n$ and m suff. large.

If X is not normal, let $\nu : X' \longrightarrow X$ be the normalization.

Set $D' = \nu^* D$.

WTS: $\kappa(D') = n \iff (*)$ holds for D.

Projection formula $\Rightarrow$ $h^0(\mathcal{O}_{X'}(mD')) = h^0(\nu_* \mathcal{O}_{X'} \otimes \mathcal{O}_X(mD))$

But we have $0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X'} \to \mathcal{F} \to 0$

$\mathcal{F}$ is supported in codim ≥ 1.

Twisting, we get $0 \to H^0(\mathcal{O}_X(mD)) \to H^0(\nu_* \mathcal{O}_{X'} \otimes \mathcal{O}_X(mD)) \to H^0(\mathcal{F}(mD)) \to \ldots$

So $h^0(\mathcal{O}_X(mD)) \leq h^0(\nu_* \mathcal{O}_{X'} \otimes \mathcal{O}_X(mD)) \leq h^0(\mathcal{O}_X(mD)) + h^0(\mathcal{F}(mD))$

But, <u>Exercise</u>: $h^0(\mathcal{F}(mD))$ grows like m^N, where $N = \dim \operatorname{Supp} \mathcal{F} \leq n-1$.

So $h^0(\mathcal{O}_X(mD))$ grows like $m^n \iff h^0(\mathcal{O}_{X'}(mD'))$ does. $\square$

<u>Ex</u>: We already saw Nef $\not\Rightarrow$ Big. Similarly, Big $\not\Rightarrow$ nef:

let $\mu : X \to \mathbb{P}^n$ be the blowup along two distinct points P, Q.

Denote H = pullback of a hyperplane
E = exceptional divisor.

Let $D = dH - rE$

Projection formula $\Rightarrow \mu_* \mathcal{O}_X(mD) = \mu_*\left(\mu^*\mathcal{O}_{\mathbb{P}}(md) \otimes \mathcal{O}_X(-rmE)\right)$

$$= \mathcal{O}_{\mathbb{P}}(md) \otimes \mathcal{I}^{mr}_{P+Q}$$

For $m >> 0$, we have

$$0 \to H^0(\mu_*\mathcal{O}_X(mD)) \to H^0(\mathcal{O}_{\mathbb{P}}(md)) \to H^0(\mathcal{O}/m_P^{mr}) \oplus H^0(\mathcal{O}/m_Q^{mr})$$

$H^0(\mathcal{O}_{\mathbb{P}}(md))$: dim $\frac{(md)^n}{n!}$ + lower deg

$H^0(\mathcal{O}/m_P^{mr})$: dim $\frac{(mr)^n}{n!}$ + lower deg

So D is big $\Longleftrightarrow \frac{(md)^n}{n!} - \frac{2(mr)^n}{n!} \geq Cm^n$ for some $C > 0$

$\Longleftrightarrow d^n > 2r^n$

However, take ℓ = proper transform of $\overline{PQ}$.

Then $D \cdot \ell = d - 2r$

so if $2r > d$, then D is not nef, i.e. $2r^n < d^n < 2^n r^n$

$\Rightarrow$ big, not nef

<u>Kodaira's Lemma</u>: D a big divisor, F effective on X.

Then $H^0(\mathcal{O}_X(mD - F)) \neq 0 \quad \forall\, m \in N(D)$ suff. large.

Pf: Let $n = \dim X$. Consider

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0$$

D big $\Rightarrow h^0(\mathcal{O}_X(mD)) \geq c \cdot m^n$, for a constant $c>0$ and $m \gg 0$.

By exercise, $h^0(\mathcal{O}_F(mD))$ grows (at most) on the order of m^{n-1}.

Thus $h^0(\mathcal{O}_X(mD)) > h^0(\mathcal{O}_F(mD))$ for $m \gg 0$.

$\Rightarrow h^0(\mathcal{O}_X(mD-F)) > 0$. $\square$

Cor: X irr., projective, D a divisor on X.

TFAE:

(i.) D is big

(ii.) If A is an ample (integral) divisor, $\exists\ m>0$, and N effective s.t.

$$mD \equiv_{num} A + N$$

(iii.) Same as (ii) but "$\exists$ an ample..."

(iv.) Same as (iii) but replace "$\equiv_{lin}$" with "$\equiv_{num}$"

Pf: (ii.) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

Assume D is big, A ample.

Take $r >> 0$ so that rA and $(r+1)A$ are effective.

Then Kodaira's lemma $\Rightarrow \exists\, m > 0$ s.t. $H^0(\mathcal{O}_X(mD - (r+1)A)) \neq 0$

Thus, $mD - (r+1)A \equiv_{lin} N'$, N' effective.

$$\Rightarrow mD \equiv_{lin} A + \underbrace{rA + N'}_{= N}$$ and N is effective.

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Now assume (iv.) holds.

Then $mD - N \equiv_{num} A$, A ample. Thus, $mD - N$ is ample.

Thus, we can find $r > 0$ an integer s.t.

$r(mD - N) \equiv_{lin} H$ very ample

$$\Rightarrow rmD \equiv_{lin} \underset{\text{v.ample}}{H} + \underset{\text{effective}}{rN}$$

Thus, $\kappa(D) \geq \kappa(H) = \dim X \Rightarrow D$ is big. $\square$

Rmk: This shows that bigness is invariant under numerical equivalence.

<u>Cor:</u> D is big $\Rightarrow$ $e(D)=1$. (i.e. every suff large multiple of D is lin. equiv. to something effective.)

<u>Pf:</u> Suppose D is big and A is ample.

By Kodaira's Lemma (and more generally!) we can take $H =$ large multiple of A

so that H is very ample and

$$H^0(H-D) \neq 0. \text{ i.e.}$$

$H - D \equiv_{lin} H'$ is effective.

Also, by the previous cor, $\exists$ $m \in \mathbb{N}(D)$ s.t.

$$mD \equiv_{lin} H + N \text{ for effective } N.$$

Thus $(m-1)D \equiv_{lin} (H-D) + N \equiv_{lin} H' + N$, which is eff.

Thus, $m-1, m \in \mathbb{N}$, so $e(D)=1$. $\square$

<u>Restrictions of big divisors</u>

Restricting a big div. doesn't necessarily preserve bigness

<u>Ex:</u> $X = Bl_p \mathbb{P}^2$. Then $H+E$ is big, but $\mathcal{O}_E(H+E) = \mathcal{O}_E(-1)$.

However, since a high multiple is "birationally very ample", restriction to a "general" subvariety is big.... i.e.

Cor: D a big div on X. Then there's some closed $V \subseteq X$ s.t. if $Y \subseteq X$ is a subvariety s.t. $Y \not\subseteq V$, then $\mathcal{O}(D)|_Y$ is big.

Pf: We can write $mD \equiv_{lin} \underset{\substack{\uparrow \\ \text{v. ample}}}{H} + \underset{\substack{\nwarrow \\ \text{effective}}}{N}$

V = support of N

If $Y \not\subseteq V$, then $\mathcal{O}_Y(mD)$ = v. ample + eff $\Rightarrow \mathcal{O}_Y(D)$ is big. $\square$

Def: A $\mathbb{Q}$-divisor D is big if there is some integer $m>0$ s.t. mD is integral and big.

Rmk: This def agrees w/ def of big for $\mathbb{Z}$-divisors, i.e. D (integral) big $\Leftrightarrow$ mD is big.

Numerical criterion for bigness:

Thm: X projective, dim n, D and E nef $\mathbb{Q}$-divisors. Suppose
$$D^n > n(D^{n-1} \cdot E).$$

Then $D-E$ is big.

Pf: Since inequality is strict, it still holds when moving D,E

ε in any direction.

Thus, can replace D and E by $D+\varepsilon H$, $E+\varepsilon H$, where H is ample, $\Rightarrow$ assume D, E are ample.
$\varepsilon > 0$ small.

Can also replace D and E by mD, mE and inequality holds $\Rightarrow$ assume D, E are integral and v. ample.

Choose $E_1, E_2, \ldots \in |E|$ general divisors. Fix $m \geq 1$.

Then $\mathcal{O}_X(m(D-E)) \cong \mathcal{O}_X(mD - E_1 - E_2 - \cdots - E_m)$

$\Rightarrow$

$0 \to H^0(\mathcal{O}_X(m(D-E))) \longrightarrow H^0(\mathcal{O}_X(mD)) \longrightarrow \bigoplus_{i=1}^{m} H^0(\mathcal{O}_{E_i}(mD))$ is exact

mD is ample $\Rightarrow \mathcal{O}_{E_i}(mD)$ is ample, so Asymptotic R-R $\Rightarrow$

$$h^0(\mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1})$$

$$h^0(\mathcal{O}_{E_i}(mD)) = \frac{E_i \cdot D^{n-1}}{(n-1)!} m^{n-1} + O(m^{n-2})$$

$E_i \equiv_{lin} E$, so $E_i D^{n-1} = E \cdot D^{n-1}$

$$\Rightarrow \bigoplus_{i=1}^{m} h^0(\mathcal{O}_{E_i}(mD)) = \frac{E \cdot D^{n-1}}{(n-1)!} m^n + O(m^{n-1})$$

$$\Rightarrow h^0(\mathcal{O}_X(m(D-E))) \geq \frac{D^n}{n!} m^n - n\left(\frac{E \cdot D^{n-1}}{(n-1)!}\right) m^n + O(m^{n-1})$$

By hypothesis, $D^n - nE \cdot D^{n-1} > 0$ so $D - E$ grows like $C \cdot m^n$, $C > 0$

so $D - E$ is big. $\square$

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<u>Thm</u>: Let X be irreducible, projective, dim n, and D a nef divisor. Then D is big $\Longleftrightarrow$ $D^n > 0$.

<u>Pf</u>: "$\Longleftarrow$": If $D^n > 0$, then setting $E = 0$, D must be big by previous thm.

"$\Longrightarrow$" Assume D is big and nef.

If $\dim X = 1$, then big + nef $\Longleftrightarrow$ ample so $D > 0$.

Assume statement holds for $\dim X \leq n-1$

Then $mD \equiv_{lin} H + N$ for $m > 0$, H very ample, N effective.

Then $mD^n = H \cdot D^{n-1} + \underbrace{N \cdot D^{n-1}}_{\substack{\vee\!| \\ 0}} \geq H \cdot D^{n-1}$

(Kleiman's Thm)

H is v. ample $\Rightarrow$ we can move it so it's not in a closed subvariety

$\Rightarrow D|_H$ is big and nef (which is preserved by restriction).

$\Rightarrow$ by induction $D^{n-1} \cdot H > 0 \Rightarrow D^n > 0$. $\square$

<u>Rmk</u>: If D is just big and not nef, then D^n can be arbitrarily negative:

<u>Ex</u>: $X = Bl_p \mathbb{P}^2$. Then H = pullback of line is nef and $H^2 = 1 > 0$, so H is nef.

So

$D = H + aE$ is big, but

$D^2 = 1 - a^2 << 0$ for $a >> 0$.

<u>Big $\mathbb{R}$-divisors</u>

$D \in Div_{\mathbb{R}}(X)$ is <u>big</u> if $D = \sum \underset{\mathbb{R}_{>0}}{a_i} \underset{\text{integral, big}}{D_i}$

<u>Exercise</u>: D, D' $\mathbb{R}$ divisors s.t. $D \equiv_{num} D'$

Show that D is big $\Leftrightarrow D'$ is big.

<u>Claim</u>: $D \in Div_{\mathbb{R}}(X)$ is big $\Leftrightarrow D \equiv_{num} \underset{\text{ample } \mathbb{R}\text{-div}}{A} + \underset{\text{effective } \mathbb{R}\text{-div}}{N}$

<u>Pf</u>: If $D = \sum \overset{>0}{a_i} \overset{\text{big}}{D_i}$ is big,

$\exists$ some $m > 0$ s.t. $mD_i \equiv A + N$

$\exists$ some $m>0$ s.t. $mD_i \equiv_{num} \underset{\text{ample}}{A_i} + \underset{\text{eff}}{N_i}$

$$\Rightarrow \quad D = \underbrace{\sum \frac{a_i}{m} A_i}_{\mathbb{R}\text{-ample}} + \underbrace{\sum \frac{a_i}{m} N_i}_{\mathbb{R}\text{-eff}} .$$

Now assume $D = \underset{\mathbb{R}\text{-ample}}{A} + \underset{\mathbb{R}\text{-eff}}{N}$

We can rewrite this as $D = \sum (a_i A_i + b_i N_i) = \sum a_i \left(A_i + \frac{b_i}{a_i} N_i\right)$

$a_i, b_i > 0$ real, and $A_i, N_i \in \mathrm{Div}_{\mathbb{Z}}(X)$ ample and eff, resp.

so it suffices to show $\underset{\mathbb{Z}\text{-ample}}{A'} + \underset{\mathbb{R}_{>0}}{s} \underset{\mathbb{Z}\text{-eff}}{N'}$ is big

We're done if $s \in \mathbb{Q}$ (Clear denominator).

Otherwise, let s_1, s_2 be positive, rat'l s.t. $s_1 < s < s_2$

Can find $t \in [0,1]$ s.t. $s = t s_1 + (1-t) s_2$

Thus, $A' + sN' = t(A' + s_1 N') + (1-t)(A' + s_2 N')$ is big.

Now, notice that if $D \equiv_{num} A + N$, then $D - N \equiv_{num} A$, so $D - N = A'$ is ample $\Rightarrow$ $D = N + A'$ is thus big by the above argument.

__Cor__: D a big $\mathbb{R}$-divisor, E an arbitrary $\mathbb{R}$-divisor.

Then $D + \varepsilon E$ is big for $\varepsilon > 0$ suff. small.

(i.e. Bigness is an open condition)

Pf: $\underset{\substack{\uparrow \\ \text{ample}}}{D \equiv_{num} A} + \underset{\substack{\uparrow \\ \text{eff}}}{N} \Rightarrow D + \varepsilon E \equiv_{num} \underbrace{(A + \varepsilon E)}_{\text{ample for small } \varepsilon.} + N$ $\square$